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Blossoms of generalized derivatives in Chebyshev spaces

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Abstract

A classical formula gives the blossom of the derivative of a polynomial function in terms of its own blossom. We extend this result to the Chebyshevian framework.

Keywords: Blossoms; Extended Chebyshev spaces; Chebyshev blossoms

1. Introduction

How to compute the Bézier points of the derivative of a polynomial function from its own Bézier points? How to compute the poles of the derivative of a spline from its own poles? Here are examples of problems the solution of which are made very elegant by using blossoms. Indeed, Bézier points as well as poles can be expressed in terms of blossoms, and the blossom of the derivative is easy to derive from the blossom of the polynomial function itself. Does there exist a similar result concerning the derivatives in the Chebyshevian framework? This is the issue we address in the present paper.

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First of all we shall briefly recall how things work in the polynomial setting. Given a polynomial function P of degree less than or equal to n, with values in $\mathbb{R}^d$, its blossom is the unique function p of n variables which is symmetric, n-affine (i.e., affine in each variable) on $\mathbb{R}^n$, and which gives P by restriction to the diagonal of $\mathbb{R}^n$, i.e.,

$$p(t^{[n]}) = P(t) \ , \quad t \in \mathbb{R}, \tag{1.1}$$

where the notation $t^{[k]}$ stands for t repeated k times. Differentiating (1.1) and using the symmetry and the multi-affinity of p, it is straightforward to calculate the derivative of P, which provides us with the following classical result [8]:

$$P'(t) = \frac{n}{y - x}\left[p(t^{[n-1]}, y) - p(t^{[n-1]}, x)\right], \tag{1.2}$$

where x, y denote any real numbers such that $x \neq y$. As a polynomial of degree less than or equal to $(n-1)$, P' has a blossom, which we denote by $p^{\{1\}}$: it is the unique function of $(n-1)$ variables which is symmetric, $(n-1)$-affine, and which gives P' by restriction to the diagonal of $\mathbb{R}^{n-1}$. Hence (1.2) readily leads to

$$p^{\{1\}}(a_1, \ldots, a_{n-1}) = \frac{n}{y - x}\left[p(a_1, \ldots, a_{n-1}, y) - p(a_1, \ldots, a_{n-1}, x)\right]. \tag{1.3}$$

Again this equality is valid for any real numbers x, y, with $x \neq y$.

The purpose of this paper is to extend formulae (1.2) and (1.3) to the Chebyshevian framework, and this will be done in Section 3. Now, in the polynomial setting, differentiation diminishes the degree, or, equivalently, the dimension of the space we are working with, hence the number of variables in the blossoms. This is no longer true in the general Chebyshevian setting, at least using the ordinary differentiation. But, as is well known, with any extended Chebyshev space it is possible to associate appropriate differential operators which, in this larger setting, play the same role as the ordinary derivatives in polynomial spaces, whence the expression "generalized derivatives" appearing in the title. A brief summary concerning these generalized derivatives is given in Section 2, in which we also recall how blossoms are defined, and how they can be calculated using the generalized derivatives.

2. Preliminaries

We briefly present here all the tools about extended Chebyshev spaces and Chebyshev blossoming which will be necessary in the following section. For further acquaintance with these subjects, the reader can, for instance, refer to [1,2,9] and [3,4,6,7], respectively.

2.1. Extended Chebyshev spaces

Let $\mathcal{E}_n$ be an $(n+1)$-dimensional space of C^∞ functions on a given interval I with a nonempty interior. Given a basis $(V_0, \dots, V_n)$ of $\mathcal{E}_n$, we set $\mathbf{V}(t) := (V_0(t), \dots, V_n(t))$ for all $t \in I$. Let us first recall that $\mathcal{E}_n$ is said to be *an extended Chebyshev space* (EC) *on* I when any of its nonzero elements vanishes at most n times on I, counting multiplicities. Equivalently, it is an EC space on I iff any system

$$\langle X, \mathbf{V}^{(k)}(b_i) \rangle = \alpha_i^k, \quad 0 \leqslant k \leqslant \mu_i - 1, \ \ 1 \leqslant i \leqslant r, \tag{2.1}$$

has a unique solution, in which $\langle \, . \, , \, . \, \rangle$ stands for the inner product of $\mathbb{R}^{n+1}$, $b_1, \dots, b_r$ are pairwise distinct elements of I, $\mu_1, \dots, \mu_r$ are positive integers such that $\sum_{i=1}^r \mu_i = n+1$, and the α_i^ks are any real numbers.

Given $n+1$ *weight functions* $w_0, \dots, w_n$ supposed to be C^∞ and positive on I, we consider the differential operators $L_0, \dots, L_n$ defined on $C^\infty(I)$ as follows:

$$L_0 V := \frac{1}{w_0} V, \quad L_i V := \frac{1}{w_i} (L_{i-1} V)', \quad i = 1, \dots, n. \tag{2.2}$$

Denoting by D the ordinary derivative operator, it is known that the $(n+1)$-dimensional space $\mathcal{E}_n := \operatorname{Ker} D \circ L_n$ is then an extended Chebyshev space on I. We shall denote it as $\mathrm{EC}(w_0, \dots, w_n)$ and we shall say that it is the extended Chebyshev space associated with $w_0, \dots, w_n$. For instance, the extended Chebyshev space associated with the weight functions $w_0 := \cdots := w_n := \mathbb{1}$ is the space $\mathcal{P}_n$ of all polynomials of degree less than or equal to n, i.e. $\mathcal{P}_n = \mathrm{EC}(\underbrace{\mathbb{1}, \dots, \mathbb{1}}_{n+1 \text{ times}})$.

A given sequence of weight functions $w_0, \dots, w_n$ actually yields a nested sequence of EC spaces on I:

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n \tag{2.3}$$

with

$$\mathcal{E}_i := \operatorname{Ker} D \circ L_i = \mathrm{EC}(w_0, \dots, w_i), \quad 0 \leqslant i \leqslant n. \tag{2.4}$$

From (2.2) it is easy to deduce that, for any $b \in I$, and any positive μ, the μ vectors $L_0\mathbf{V}(b), \dots, L_{\mu-1}\mathbf{V}(b)$ span the same linear subspace as the μ vectors $\mathbf{V}(b), \dots, \mathbf{V}^{(\mu-1)}(b)$. Therefore, under the same assumptions as in (2.1), any system

$$\langle X, L_k\mathbf{V}(b_i) \rangle = \alpha_i^k, \quad 0 \leqslant k \leqslant \mu_i - 1, \ \ 1 \leqslant i \leqslant r, \tag{2.5}$$

has a unique solution too. In this situation, also of interest are the following spaces:

$$\mathcal{E}_n^{\{i\}} := L_i \mathcal{E}_n := \{L_i V, \ V \in \mathcal{E}_n\}, \quad 0 \leqslant i \leqslant n. \tag{2.6}$$

Due to the recursive definition of the operators $L_0, \dots, L_n$, we clearly have

$$\mathcal{E}_n^{\{i\}} = \mathrm{EC}(\mathbb{1}, w_{i+1}, \dots, w_n), \quad 0 \leqslant i \leqslant n. \tag{2.7}$$

Hence, (2.6) provides another (nonnested) sequence of EC spaces on the same interval I.

Note that, conversely, if the interval I is closed and bounded, any $(n+1)$-dimensional extended Chebyshev space on I, is the space $\mathrm{EC}(w_0, \ldots, w_n)$ for some weight functions $w_0, \ldots, w_n$ (see [7]).

2.2. Blossoms in extended Chebyshev spaces

It is known that blossoms exist in any extended Chebyshev space. However, defining them in the most general extended Chebyshev spaces requires to work in projective spaces (see [7]). In order to obtain blossoms in the affine setting, we have to particularize a little the spaces we are working with. For this reason, from now on we shall consider a space $\mathcal{E}_n$ of C^∞ functions on the interval I containing the constants and such that the space $D\mathcal{E}_n$ is an n-dimensional extended Chebyshev space on I [3,4,6]. The space $\mathcal{E}_n$ itself is then automatically an $(n+1)$-dimensional extended Chebyshev space on I.

Select a basis $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ in $\mathcal{E}_n$, and set $\Phi := (\Phi_1, \ldots, \Phi_n) : I \to \mathbb{R}^n$. The osculating flat of any order $i \geqslant 0$ at a point $t \in I$ is the affine flat going through $\Phi(t)$ and the direction of which is the linear space spanned by $\Phi'(t), \ldots, \Phi^{(i)}(t)$. We denote it by $\mathrm{Osc}_i\, \Phi(t)$. For any $i \leqslant n$ its dimension is equal to i. In particular, $\mathrm{Osc}_0\, \Phi(t) = \{\Phi(t)\}$.

Given any distinct points $\tau_1, \ldots, \tau_r \in I$ and any positive integers $\mu_1, \ldots, \mu_r$ the sum of which is equal to n, the intersection of the r osculating flats $\mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i)$, $1 \leqslant i \leqslant r$, consists of a single point. The blossom of Φ is then the function $\varphi := (\varphi_1, \ldots, \varphi_n) : I^n \to \mathbb{R}^n$ defined by setting:

$$\{\varphi(t_1, \ldots, t_n)\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i), \tag{2.8}$$

whenever $(t_1, \ldots, t_n) \in I^n$ is equal to $(\tau_1^{[\mu_1]}, \ldots, \tau_r^{[\mu_r]})$ up to a permutation. It is a symmetric function and it satisfies the diagonal property

$$\varphi(t^{[n]}) = \Phi(t) \quad \text{for all } t \in I. \tag{2.9}$$

The third interesting property of the blossom φ is that it is *pseudoaffine* with respect to each variable. Let us explain what this means. For any $(a_1, \ldots, a_{n-1}) \in I^{n-1}$, equal to $(b_1^{[\mu_1]}, \ldots, b_r^{[\mu_r]})$ up to a permutation, with pairwise distinct b_i's and positive μ_i's, the affine flat $\mathcal{D} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(b_i)$ is an affine line. The function $\varphi(a_1, \ldots, a_{n-1}, \cdot) : I \to \mathbb{R}^n$ takes its values in the line $\mathcal{D}$ and it is strictly monotone on I.

Blossoms in the space $\mathcal{E}_n$ are then defined as affine images of φ, independently of the initial choice of the "mother" function Φ. For instance φ_i is the blossom of Φ_i for $1 \leqslant i \leqslant n$.

2.3. How to compute blossoms?

We need to say more about how to obtain the blossoms, that is about how to find the function φ. Let us first observe that in order to calculate the value of the blossom φ of Φ at a given n-tuple $(t_1, \ldots, t_n) \in I^n$, we can limit ourselves to considering a closed bounded subinterval J of I containing the t_i's, and with a nonempty interior. The restriction of $D\mathcal{E}_n|J$ of the space $D\mathcal{E}_n$ to J can be written as $D\mathcal{E}_n|J = \mathrm{EC}(w_1, \dots, w_n)$ for some weight functions $w_1, \dots, w_n$ defined on J, and therefore $\mathcal{E}_n|J = \mathrm{EC}(\mathbb{1}, w_1, \dots, w_n)$.

For this reason, without loss of generality we shall assume in the rest of the paper that $\mathcal{E}_n := \mathrm{EC}(\mathbb{1}, w_1, \dots, w_n)$ with fixed weight functions $w_1, \dots, w_n$ on I. For any $t \in I$, and any $i \leqslant n$, the direction of the osculating flat $\mathrm{Osc}_i\, \Phi(t)$ is then spanned by the i vectors $L_1\Phi(t), \dots, L_i\Phi(t)$ as well, where the operators $L_1, \dots, L_n$ are still defined by (2.2), but with now $w_0 := \mathbb{1}$, that is $L_0 = Id$. In particular, the direction orthogonal to the affine hyperplane $\mathrm{Osc}_{n-1}\, \Phi(t)$ is given by the vector

$$L_1\Phi(t) \wedge \cdots \wedge L_{n-1}\Phi(t),$$

where, for any $V_1, \dots, V_{n-1} \in \mathbb{R}^n$, we denote by $V_1 \wedge \cdots \wedge V_{n-1}$ the unique vector which represents the linear functional $X \in \mathbb{R}^n \mapsto \det(X, V_1, \dots, V_{n-1})$. Note that, due to (2.2) and to the equality $\mathcal{E}_n = \mathrm{Ker}\, D \circ L_n$, we have

$$DL_i\Phi = w_{i+1}L_{i+1}\Phi, \quad i = 0, \dots, n-1, \quad DL_n\Phi = 0. \tag{2.10}$$

As an immediate consequence, the number

$$\delta := \det(L_1\Phi(t), \dots, L_n\Phi(t)) \tag{2.11}$$

does not depend on the point $t \in I$. To be convinced of this, just differentiate the right-hand side of (2.11) and use (2.10). For any $t \in I$, let us introduce the following n linearly independent vectors:

$$\Phi^{[k]}(t) := \frac{1}{\delta} L_1\Phi(t) \wedge \cdots \wedge L_{k-1}\Phi(t) \wedge L_{k+1}\Phi(t) \wedge \cdots \wedge L_n\Phi(t), \quad 1 \leqslant k \leqslant n. \tag{2.12}$$

They satisfy

$$\langle L_j\Phi(t), \Phi^{[k]}(t) \rangle = (-1)^{k-1}\ \delta_{j,k}, \quad t \in I, \ 1 \leqslant j, k \leqslant n. \tag{2.13}$$

Another immediate consequence of (2.10) is that

$$D\Phi^{[k]}(t) = w_k(t)\Phi^{[k-1]}(t), \quad 2 \leqslant k \leqslant n, \quad D\Phi^{[1]}(t) = 0, \quad t \in I. \tag{2.14}$$

Consider the weight functions

$$\widehat{w}_i := w_{n+1-i}, \quad 1 \leqslant i \leqslant n-1 \tag{2.15}$$

and the corresponding differential operators on $C^\infty(I)$:

$$\widehat{L}_0 V := V, \quad \widehat{L}_i V := \frac{1}{\widehat{w}_i}(\widehat{L}_{i-1}V)', \quad i = 1, \dots, n-1. \tag{2.16}$$

We can then write the equalities (2.14) as

$$\widehat{L}_k\Phi^{[n]} = \Phi^{[n-k]}, \quad 1 \leqslant k \leqslant n-1, \quad D\widehat{L}_{n-1}\Phi^{[n]} = 0. \tag{2.17}$$

Accordingly, the n components $\Phi_1^{[n]}, \dots, \Phi_n^{[n]}$ of $\Phi^{[n]}$ are linearly independent on I, and they span the n-dimensional extended Chebyshev space $\widehat{\mathcal{E}}_n := \mathrm{EC}(\mathbb{1}, w_n, w_{n-1}, \dots, w_2)$.

We know that, for any $t \in I$, the vector $\Phi^{[n]}(t)$ spans the direction orthogonal to the osculating hyperplane $\mathrm{Osc}_{n-1}\, \Phi(t)$. Consequently, for pairwise distinct $t_1, \dots, t_n \in I$, the calculation of the point $X := (X_1, \ldots, X_n) := \varphi(t_1, \ldots, t_n)$ is done by solving the linear system:

$$\langle X, \Phi^{[n]}(t_i) \rangle = \langle \Phi(t_i), \Phi^{[n]}(t_i) \rangle, \quad i = 1, \ldots, n. \tag{2.18}$$

More generally, according to (2.13), for $1 \leqslant i \leqslant n$, each vector $\Phi^{[n-r+1]}(t) = \widehat{L}_{r-1}\Phi^{[n]}(t)$, $1 \leqslant r \leqslant i$, is orthogonal to the $(n-i)$ vectors $L_1\Phi(t), \ldots, L_{n-i}\Phi(t)$ which span the direction of $\mathrm{Osc}_{n-i}\,\Phi(t)$. Hence, the i linearly independent vectors $\Phi^{[n-i+1]}(t), \ldots, \Phi^{[n]}(t)$ form a basis of the linear subspace orthogonal to the direction of $\mathrm{Osc}_{n-i}\,\Phi(t)$. On this account, if now $(t_1, \ldots, t_n) = (\tau_1^{[\mu_1]}, \ldots, \tau_r^{[\mu_r]})$, for pairwise distinct τ_is and positive μ_is, the calculation of $X := (X_1, \ldots, X_n) := \varphi(t_1, \ldots, t_n)$ must be done by solving the system:

$$\langle X, \Phi^{[n-k]}(\tau_i) \rangle = \langle \Phi(\tau_i), \Phi^{[n-k]}(\tau_i) \rangle, \quad 0 \leqslant k \leqslant \mu_i - 1, \quad 1 \leqslant i \leqslant r. \tag{2.19}$$

3. Generalized derivative formula

In this section, we assume that $\mathcal{E}_n = \mathrm{EC}(\mathbb{1}, w_1, \ldots, w_n)$, with $n \geqslant 2$. We now address the problem of calculating the blossoms in the n-dimensional extended Chebyshev space $\mathcal{E}_n^{\{1\}} = L_1\mathcal{E}_n := \{L_1 V,\ V \in \mathcal{E}_n\}$ introduced in (2.6), from those in the space $\mathcal{E}_n$. From now on, U will stand for the following function

$$U(t) := \int_a^t w_1(\xi)\,d\xi, \tag{3.1}$$

where a is any fixed point in I. Note that $(\mathbb{1}, U)$ is a basis of the space $\mathcal{E}_1 = \mathrm{EC}(\mathbb{1}, w_1)$ introduced in (2.3). Due to the nestedness of the sequence (2.3), this function U is also an element of the space $\mathcal{E}_n$.

The following theorem is the central result of the present paper. It gives the blossom of the generalized derivative of any element of the space $\mathcal{E}_n$, and it is the proper extension of formula (1.3) to the Chebyshevian framework.

Theorem 1. *Let u denote the blossom of the function U defined in* (3.1) *viewed as an element of $\mathcal{E}_n$. Given a function $F \in \mathcal{E}_n$, with blossom f, the blossom $f^{\{1\}}$ of the function $L_1F \in \mathcal{E}_n^{\{1\}}$ is then given by*:

$$f^{\{1\}}(a_1, \ldots, a_{n-1}) := \frac{f(a_1, \ldots, a_{n-1}, y) - f(a_1, \ldots, a_{n-1}, x)}{u(a_1, \ldots, a_{n-1}, y) - u(a_1, \ldots, a_{n-1}, x)}, \tag{3.2}$$

where x, y are any two distinct points in I. Hence, *in particular*,

$$L_1F(t) = \frac{f(t^{[n-1]}, y) - f(t^{[n-1]}, x)}{u(t^{[n-1]}, y) - u(t^{[n-1]}, x)}, \quad t \in I. \tag{3.3}$$

Before proving the latter theorem, let us first check that formulae (3.2) and (3.3) are indeed direct extensions of the polynomial ones (1.3) and (1.2), respectively. In the polynomial setting, $w_1 = \mathbb{1}$ and with $a = 0$, the corresponding function U is $U(t) := t$. The blossom u of U viewed as an element of $\mathcal{E}_n = \mathcal{P}_n$ is given by

$$u(a_1, \dots, a_n) := \frac{a_1 + \cdots + a_n}{n}, \quad a_1, \dots, a_n \in \mathbb{R}.$$

Hence, $u(a_1, \dots, a_{n-1}, y) - u(a_1, \dots, a_{n-1}, x) = (y - x)/n$.

As pointed out by formula (3.2), the blossom u of U plays a fundamental role in calculating the blossoms in the space $\mathcal{E}_n^{\{1\}}$. We shall first focus on this function u. Formula (3.2) cannot be consistent without the function $u(a_1, \dots, a_{n-1}, \cdot)$ being one-to-one on I. We shall more precisely prove the following result.

Proposition 2. *For any $a_1, \dots, a_{n-1} \in I$, the function $u(a_1, \dots, a_{n-1}, \cdot)$ is strictly increasing on I.*

Let us first establish the following lemma.

Lemma 3. *Given $x, y \in I$, with $x < y$, let*

$$p_k(x, y) := u(x^{[n-k]}, y^{[k]}), \quad 0 \leqslant k \leqslant n, \tag{3.4}$$

denote the Chebyshev–Bézier points of U (viewed as an element of $\mathcal{E}_n$) w.r.t. (x, y). They satisfy:

$$p_0(x, y) < p_1(x, y) < \cdots < p_n(x, y). \tag{3.5}$$

Proof. Thanks to the nestedness of the sequence (2.3), for a given i, $1 \leqslant i \leqslant n$, the function U can also be viewed as an element of the $(i+1)$-dimensional EC space $\mathcal{E}_i = \mathrm{EC}(\mathbb{1}, w_1, \dots, w_i)$, $1 \leqslant i \leqslant n$. As so, it possesses a blossom, which is a function of i variables. The values of this blossom at the i-tuples $(x^{[i-k]}, y^{[k]})$, $0 \leqslant k \leqslant i$, are the Chebyshev–Bézier points (w.r.t. (x, y)) of the function U viewed as an element of $\mathcal{E}_i$. Let us denote them

$$p_k^i(x, y), \quad 0 \leqslant k \leqslant i.$$

In particular, $p_0^1(x, y) = U(x)$, $p_1^1(x, y) = U(y)$, and $p_k^n(x, y) = p_k(x, y)$ for $0 \leqslant k \leqslant n$. We just have to prove, by induction on i, that

$$p_0^i(x, y) < p_1^i(x, y) < \cdots < p_i^i(x, y), \tag{3.6}$$

for $1 \leqslant i \leqslant n$. For $i = 1$, (3.6) simply results from the fact that U is strictly increasing on I. On the other hand, it is known that, for $1 \leqslant i \leqslant n-1$, the points $p_k^{i+1}(x, y)$, $0 \leqslant k \leqslant i+1$, can be obtained from the points $p_k^i(x, y)$, $0 \leqslant k \leqslant i$, by a *dimension elevation process*, as follows (see [5,7]):

$$p_0^{i+1}(x, y) = p_0^i(x, y),$$

$$p_k^{i+1}(x, y) = (1 - \alpha_k^i) p_{k-1}^i(x, y) + \alpha_k^i p_k^i(x, y) \quad \text{for } 1 \leqslant k \leqslant i,$$

$$p_{i+1}^{i+1}(x, y) = p_i^i(x, y)$$

with $0<\alpha_k^i<1$ for $1\leqslant k\leqslant i$. Hence, if (3.6) is satisfied for a given i, $1\leqslant i\leqslant n-1$, it is automatically satisfied for $i+1$ too. $\square$

Proof of Proposition 2. Choose a basis $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ of the space $\mathcal{E}_n$ such that $\Phi_1 = U$, and therefore $u = \varphi_1$. Suppose that $(a_1, \ldots, a_{n-1}) = (b_1^{[\mu_1]}, \ldots, b_r^{[\mu_r]})$ with pairwise distinct $b_1, \ldots, b_r \in I$, and positive integers $\mu_1, \ldots, \mu_r$. From our recollections in Section 2.2, we know that the point $\varphi(a_1, \ldots, a_{n-1}, t)$ moves in a strictly monotonic way on the affine line $\bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i}\, \Phi(b_i)$. Therefore, each component $\varphi_k(a_1, \ldots, a_{n-1}, \cdot)$ of $\varphi(a_1, \ldots, a_{n-1}, \cdot)$ is either constant on I, or strictly increasing on I, or strictly decreasing on I. Let us show that the first one, $u(a_1, \ldots, a_{n-1}, \cdot)$, is strictly increasing.

(1) *Let us first prove that $u(a_1, \ldots, a_{n-1}, \cdot)$ is one-to-one on I*: Given two points $x \neq y$ in I, let us set

$$X := (X_1, \ldots, X_n) := \varphi(a_1, \ldots, a_{n-1}, x),$$
$$Y := (Y_1, \ldots, Y_n) := \varphi(a_1, \ldots, a_{n-1}, y). \tag{3.7}$$

We thus have

$$X_1 = \varphi_1(a_1, \ldots, a_{n-1}, x) = u(a_1, \ldots, a_{n-1}, x),$$
$$Y_1 = \varphi_1(a_1, \ldots, a_{n-1}, y) = u(a_1, \ldots, a_{n-1}, y). \tag{3.8}$$

We know that $X \neq Y$ and what we actually have to prove is that $X_1 \neq Y_1$. Now, according to (2.8), whether or not $x, y \in \{b_1, \ldots, b_r\}$, the two points X and Y belong to $\mathrm{Osc}_{n-\mu_i}\, \Phi(b_i)$, $1\leqslant i\leqslant r$. By (2.19), we thus have

$$\begin{aligned}\langle X, \Phi^{[n-k]}(b_i)\rangle &= \langle Y, \Phi^{[n-k]}(b_i)\rangle \\ &= \langle \Phi(b_i), \Phi^{[n-k]}(b_i)\rangle, \quad 0\leqslant k\leqslant \mu_i - 1,\ \ 1\leqslant i\leqslant r.\end{aligned}$$

Hence

$$\langle Y - X, \Phi^{[n-k]}(b_i)\rangle = 0, \quad 0\leqslant k\leqslant \mu_i - 1,\ \ 1\leqslant i\leqslant r,$$

i.e., on account of (2.17)

$$\sum_{j=2}^{n} (Y_j - X_j)\widehat{L}_k \Phi_j^{[n]}(b_i) = (X_1 - Y_1)\widehat{L}_k \Phi_1^{[n]}(b_i), \quad 0\leqslant k\leqslant \mu_i - 1,\ \ 1\leqslant i\leqslant r. \tag{3.9}$$

From (2.17) and (2.14), we know that

$$D\widehat{L}_{n-2}\Phi^{[n]} = D\Phi^{[2]} = w_2\Phi^{[1]}. \tag{3.10}$$

Now, due to our choice of $\Phi_1 = U$, the first component of each function $L_2\Phi, \ldots, L_n\Phi$ is equal to 0. It therefore results from our definition (2.12) that all components $\Phi_k^{[1]}$ of $\Phi^{[1]}$ are zero, except for that of index 1. Accordingly, (3.10) proves that the functions $\Phi_2^{[n]}, \ldots, \Phi_n^{[n]}$ belong to $\mathrm{Ker}\, D \circ \widehat{L}_{n-2}$, hence form a basis of the $(n-1)$-dimensional space $\mathrm{EC}(\mathbb{1}, w_n, \ldots, w_3)$. Accordingly, considered as a system with unknowns the $(Y_k - X_k)$, $k = 2, \ldots, n$, the $(n-1)$ equalities (3.9) provide a unique solution. Hence, $Y_1 = X_1$ would imply $Y_k = X_k$ for all indices k, that is $Y = X$, which is wrong. Therefore, $Y_1 \neq X_1$ is proved.

(2) *Let us now prove that $u(a_1, \dots, a_{n-1}, \cdot)$ is strictly increasing on I*: Using the same notations as in the previous part of the proof, we just have to show that, a relevant choice of $x < y$ guarantees that $X_1 = u(a_1, \dots, a_{n-1}, x) < Y_1 = u(a_1, \dots, a_{n-1}, y)$.

Select $x < y$ in I so that $x \leqslant \min(a_1, \dots, a_{n-1})$, and $y \geqslant \max(a_1, \dots, a_{n-1})$. For $0 \leqslant i \leqslant n-1$, let us introduce the real numbers

$$q_k^i := u(a_1, \dots, a_i, x^{[n-i-k]}, y^{[k]}), \quad 0 \leqslant k \leqslant n-i.$$

Since $X_1 = q_0^{n-1}$, and $Y_1 = q_1^{n-1}$, it is sufficient to show that, for any $i \leqslant n-1$, the sequence q_k^i, $0 \leqslant k \leqslant n-i$, is strictly increasing. This will be done by induction on i. For $i = 0$, this follows from Lemma 3, for the points q_k^0, $0 \leqslant k \leqslant n$, are nothing but the Chebyshev–Bézier points $p_k(x, y)$, $0 \leqslant k \leqslant n$, of U w.r.t. (x, y) introduced in (3.4).

Let us assume that the result is proved for a given integer i, $0 \leqslant i \leqslant n-2$. If $a_{i+1} = x$ (resp. $a_{i+1} = y$), then we have $q_k^{i+1} = q_k^i$ (resp. $q_k^{i+1} = q_{k+1}^i$) for $0 \leqslant k \leqslant n-i-1$, and there is nothing to prove. Suppose now that $x < a_{i+1} < y$. The pseudoaffinity of φ implies that, for $0 \leqslant k \leqslant n-i-1$, the number q_k^{i+1} is a strictly convex combination of q_k^i and q_{k+1}^i. The desired result is thus proved for $i+1$. $\square$

Proof of Theorem 1. Again, we choose a basis $(\mathbb{1}, \Phi_1, \dots, \Phi_n)$ of $\mathcal{E}_n$ so that $\Phi_1 = U$. Then, the functions $L_1\Phi_1 = \mathbb{1}, L_1\Phi_2, \dots, L_1\Phi_n$ form a basis of the space $\mathcal{E}_n^{\{1\}}$. Let us set

$$\Phi^{\{1\}} := (L_1\Phi_2, \dots, L_1\Phi_n). \tag{3.11}$$

We actually intend to determine the blossom $\varphi^{\{1\}} : I^{n-1} \to \mathbb{R}^{n-1}$ of the function $\Phi^{\{1\}}$ using the blossom φ of Φ. Suppose again that $(a_1, \dots, a_{n-1}) = (b_1^{[\mu_1]}, \dots, b_r^{[\mu_r]})$ with pairwise distinct $b_1, \dots, b_r \in I$, and positive integers $\mu_1, \dots, \mu_r$ with sum equal to $n-1$. We know that the value of $\varphi^{\{1\}}$ at $(a_1, \dots, a_{n-1})$ is given by

$$\{\varphi^{\{1\}}(a_1, \dots, a_{n-1})\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-1-\mu_i}\, \Phi^{\{1\}}(b_i). \tag{3.12}$$

Given $x \neq y$ any two points in I, let us use the notations introduced in (3.7). Since $x \neq y$, Proposition 2 ensures that $X_1 \neq Y_1$. Setting

$$X^{\{1\}} := \left(\frac{Y_2 - X_2}{Y_1 - X_1}, \dots, \frac{Y_n - X_n}{Y_1 - X_1}\right), \tag{3.13}$$

we actually have to prove that

$$X^{\{1\}} = \varphi^{\{1\}}(a_1, \dots, a_{n-1}). \tag{3.14}$$

Equality (3.2) will then follow via affine maps. On account of (3.12), in order to prove (3.14), we just have to check that the point $X^{\{1\}}$ defined in (3.13) belongs to each osculating flat $\mathrm{Osc}_{n-1-\mu_i}\, \Phi^{\{1\}}(b_i)$, $1 \leqslant i \leqslant r$.

Let us consider a given i, $1 \leqslant i \leqslant r$. Whether or not x or y are equal to b_i, the two points X and Y belong to the osculating flat $\mathrm{Osc}_{n-\mu_i}\,\Phi(b_i)$. Hence, there exist (unique) numbers $\lambda_1^i(x), \ldots, \lambda_{n-\mu_i}^i(x)$, and $\lambda_1^i(y), \ldots, \lambda_{n-\mu_i}^i(y)$ such that

$$X = \Phi(b_i) + \sum_{k=1}^{n-\mu_i} \lambda_k^i(x) L_k \Phi(b_i), \quad Y = \Phi(b_i) + \sum_{k=1}^{n-\mu_i} \lambda_k^i(y) L_k \Phi(b_i).$$

The latter two equalities lead to

$$Y - X = \sum_{k=1}^{n-\mu_i} [\lambda_k^i(y) - \lambda_k^i(x)] L_k \Phi(b_i). \tag{3.15}$$

Since $L_1 \Phi_1 = \mathbb{1}$, and $L_k \Phi_1 = 0$ for $k > 1$, looking at the first components of the two sides of (3.15), we obtain

$$\lambda_1^i(y) - \lambda_1^i(x) = Y_1 - X_1. \tag{3.16}$$

Consequently, we can write the equality (3.15) as follows:

$$\frac{1}{Y_1 - X_1}\,(Y - X) = L_1 \Phi(b_i) + \sum_{k=2}^{n-\mu_i} \frac{\lambda_k^i(y) - \lambda_k^i(x)}{Y_1 - X_1} L_k \Phi(b_i). \tag{3.17}$$

Denote by $L_0^{\{1\}} = Id, L_1^{\{1\}}, \ldots, L_{n-1}^{\{1\}}$ the differential operators associated to the weights $\mathbb{1}, w_2, \ldots, w_n$, so that $\mathcal{E}_n^{\{1\}} = \mathrm{EC}(\mathbb{1}, w_2, \ldots, w_n) = \mathrm{Ker}\, D \circ L_{n-1}^{\{1\}}$. For $1 \leqslant k \leqslant n$, we then have $L_k \Phi = L_{k-1}^{\{1\}} L_1 \Phi$. Accordingly, (3.17) yields

$$X^{\{1\}} = \Phi^{\{1\}}(b_i) + \sum_{k=1}^{n-1-\mu_i} \frac{\lambda_{k+1}^i(y) - \lambda_{k+1}^i(x)}{Y_1 - X_1} L_k^{\{1\}} \Phi^{\{1\}}(b_i). \tag{3.18}$$

The equality (3.18) proves that the point $X^{\{1\}}$ belongs to $\mathrm{Osc}_{n-1-\mu_i}\,\Phi^{\{1\}}(b_i)$. $\square$

Note that exploiting better the Eq. (3.9) would give another proof of (3.2). We preferred a more explicit one.

To conclude, let us comment on Theorem 1. Formula (3.3), which we mentioned as a particular case of (3.2), is in fact elementary to obtain directly. Indeed, given distinct $x, y \in I$, the two points $\varphi(t^{[n-1]}, x)$ and $\varphi(t^{[n-1]}, y)$ are distinct (this results from the pseudoaffinity property satisfied by φ) and both lie in $\mathrm{Osc}_1\,\Phi(t)$. This ensures the existence of a nonzero real number $\lambda(t)$ such that

$$\varphi(t^{[n-1]}, y) - \varphi(t^{[n-1]}, x) = \lambda(t) L_1 \Phi(t). \tag{3.19}$$

As in the proof of Theorem 1, suppose that $\Phi_1 = U$. Then, looking at the first components in (3.19), we obtain

$$\lambda(t) = u(t^{[n-1]}, y) - u(t^{[n-1]}, x) \neq 0.$$

Accordingly, (3.19) yields

$$L_1 \Phi(t) = \frac{\varphi(t^{[n-1]}, y) - \varphi(t^{[n-1]}, x)}{u(t^{[n-1]}, y) - u(t^{[n-1]}, x)}, \tag{3.20}$$

which leads to (3.3) via affine maps. In a $(k+1)$-dimensional space it is usual to derive relations for blossoms from the corresponding relations for functions by "blossoming" $t^{[k]}$ into $t_1, \ldots, t_k$. This is exactly what we did in the polynomial case to deduce (1.3) from (1.2). What is surprising here is that, in order to transform (3.3) into (3.2), we have to"blossom" $t^{[n-1]}$ both in the numerator and the denominator of (3.3), which is hidden in the polynomial case. Note the latter direct proof of (3.20) (hence of (3.3)) does not save us having to prove (3.2) as we did.

4. Applications

4.1. Chebyshev–Bézier points

Choose two points $x, y \in I$, with $x \neq y$. Let $P_i(x,y) := f(x^{[n-i]}, y^{[i]})$, $0 \leqslant i \leqslant n$, be the Chebyshev–Bézier points of F w.r.t. (x,y). Then, applying equality (3.2) with $(a_1, \ldots, a_{n-1}) = (x^{[n-1-i]}, y^{[i]})$, we obtain the Chebyshev–Bézier points of $L_1 F$ w.r.t. (x,y), that is the points $P_i^{\{1\}}(x,y) := f^{\{1\}}(x^{[n-1-i]}, y^{[i]})$, $0 \leqslant i \leqslant n-1$, as follows:

$$P_i^{\{1\}}(x,y) = \frac{P_{i+1}(x,y) - P_i(x,y)}{p_{i+1}(x,y) - p_i(x,y)}, \quad 0 \leqslant i \leqslant n-1, \tag{4.1}$$

where as in (3.4), $p_0(x,y), \ldots, p_n(x,y)$ are the Chebyshev–Bézier points (w.r.t. (x,y)) of the function U viewed as an element of $\mathcal{E}_n$. Note that the denominators are positive if $x < y$, and negative otherwise. Taking account of (2.2), the particular case $i = 0$ gives

$$F'(x) = w_1(x) \frac{P_1(x,y) - P_0(x,y)}{p_1(x,y) - p_0(x,y)} = w_1(x) \frac{f(x^{[n-1]}, y) - F(x)}{u(x^{[n-1]}, y) - U(x)}. \tag{4.2}$$

4.2. Iterative differentiation

We shall conclude the paper by observing that iteration of (3.2) enables the calculation of the blossom $f^{\{i\}}$ of the generalized derivative $L_i F \in \mathcal{E}_n^{\{i\}}$, $1 \leqslant i \leqslant n-1$, of any element $F \in \mathcal{E}_n$ (note that $\mathcal{E}_n^{\{n\}}$ contains only constants). With this in view, generalizing (3.1), we first need to introduce the functions

$$U_i(t) := \int_a^t w_i(\xi)\, d\xi, \quad 1 \leqslant i \leqslant n, \tag{4.3}$$

so that the function U is now renamed U_1. For a given i, $(\mathbb{1}, U_i)$ is a basis of the space $\mathrm{EC}(\mathbb{1}, w_i)$ which is a subspace of the $(n-i+2)$-dimensional space $\mathcal{E}_n^{\{i-1\}} = \mathrm{EC}(\mathbb{1}, w_i, \ldots, w_n)$. This allows us to consider the following function:

$$u_i := \text{blossom of the function } U_i \text{ viewed as an element of } \mathcal{E}_n^{\{i-1\}}. \tag{4.4}$$

It is a function of $(n-i+1)$ variables. In particular, $u_1 = u$, and $u_n = U_n$. Given any $\mathcal{T}_i \in I^{n-i}$, the value of $f^{\{i\}}$ at $\mathcal{T}_i$ can be expressed as

$$f^{\{i\}}(\mathcal{T}_i) = \sum_{k=o}^{i} A_k^i(\mathcal{T}_i) f(\mathcal{T}_i, y^{[i-k]}, x^{[k]}),$$

where the coefficients $A_k^i(\mathcal{T}_i)$ can be calculated recursively according to the following formula:

$$A_k^i(\mathcal{T}_i) = \frac{A_k^{i-1}(\mathcal{T}_i, y) - A_{k-1}^{i-1}(\mathcal{T}_i, x)}{u_i(\mathcal{T}_i, y) - u_i(\mathcal{T}_i, x)}, \quad 0 \leqslant k \leqslant i,$$

with the convention that, for any $i \leqslant n-1$ and any $\mathcal{T}_i \in I^{n-i}$, $A_k^i(\mathcal{T}_i) := 0$ if $k \notin \{0, \ldots, i\}$, and with $A_0^0(\mathcal{T}_0) := 1$ for any $\mathcal{T}_0 \in I^n$.

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